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Cherenkov Radiation Spectrum

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A new method is proposed for the construction of the theoretical Cherenkov spectrum. The expression representing the spectral energy distribution has the same form as the classical Frank–Tamm formula, but the square of the index of refraction is replaced by the real part of a dielectric function, which we interpret as a function characteristic of the polarized medium in the immediate vicinity of the passing charged particle.

1. INTRODUCTION

Cherenkov radiation (CR) is usually treated as occurring when the velocity of a charged particle in a transparent medium exceeds the phase velocity of light in that medium, and is observed at a specific Cherenkov cone angle (Jelley, 1958; Zrelov, 1968). This radiation has a continuous spectrum, and the measured intensities in the visible region are in good agreement with the classical theoretical prediction (see, e.g., Collins and Reiling, 1938; Rich *et al.*, 1953). The theory states that the radiation intensity is a linear function of the frequency (UV divergence), and the theoretical upper limit of the spectrum is an old open question. The quantum mechanical treatment based on the assumption that the effect takes place in a continuous medium, whose optical properties can be expressed in terms of a refractive index or dielectric constant, $n(\omega)=[\varepsilon(\omega)]^{1/2}$, also leads to the classical result (Zrelov, 1968; Marmier and Sheldon, 1969; Ahlen, 1980).

One of the most interesting approaches to the CR was proposed by Budini (1953), who used a complex dielectric function, introduced the condition $\beta^2 \operatorname{Re} \varepsilon(\omega) > 1$ for the formation of the CR, and obtained a modified

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expression for the radiated energy, which reduces to the classical Frank–Tamm expression in the ideal case of a perfectly transparent medium [Im $\varepsilon(\omega)=0$]. A brief review of this theory is given below in Section 3.

The goal of the present paper is to obtain a theoretical spectrum of the CR which gives decreasing intensities in the near-UV region, with predictable upper limit as a function of particle velocity. In Section 4 we write an equation for the threshold velocity by using the above inequality with an entirely different physical meaning, and we plot a spectrum with an expression in which Re $\varepsilon(\omega)$ is used instead of $n^2(\omega)$ in the classical formula. This expression is derived in Section 5. Final remarks are made in Section 6. Some general properties of the dielectric function are reviewed in the Appendix. In the beginning the classical theory of Frank and Tamm is summarized.

2. CLASSICAL THEORY

The theoretical discussion of CR is found in reviews on the Cherenkov effect and its application (Jelley, 1958; Zrelov, 1968), or in textbooks on electromagnetic theory (e.g., Landau and Lifshitz, 1960; Jackson, 1975) and quantum mechanics (e.g., Schiff, 1955). A beautiful review of the classical and quantum theory, with applications in particle detection methods (Cherenkov detectors), can be found in the book of Marmier and Sheldon (1969).

In the classical investigations the Cherenkov effect is treated as the radiation produced by an electron passing through a nonmagnetic ($\mu=1$) medium of dielectric constant $\varepsilon(\omega)$ {or refractive index $n(\omega)=[\varepsilon(\omega)]^{1/2}$}. The radiated energy per unit path length is given by the Frank–Tamm formula

$$\frac{dW}{dx}=\frac{e^2}{c^2}\int\left(1-\frac{1}{\beta^2 n^2(\omega)}\right)\omega\, d\omega \tag{1}$$

where e is the electron charge, $\beta=v/c$ (v is the electron velocity, c is the velocity of light in vacuum), ω is the frequency of the emitted light, and the integral has to be calculated over frequencies with respect to which the following inequality holds:

$$\beta n(\omega)>1 \tag{2}$$

The index of refraction is usually taken to be

$$n^2(\omega)=1+\frac{a}{\omega_0^2-\omega^2} \tag{3}$$

(Jelley, 1958; Zrelov, 1968; Jackson, 1975), where $a=4\pi Ne^2/m$, N represents the number of atoms per unit volume, ω_0 is the eigenfrequency of atomic oscillators, and m is the electron mass. Note that from the inequality (2), by using equation (3), the upper limit of the integral in equation (1) can be ω_0, but the most essential and experimentally well-established feature of CR, namely the existence of a threshold velocity ($\beta_{\text{thr}} = 1/n$, if n is taken as a constant) below which there is no radiation, is disturbed.

Equation (1) gives not only the radiated energy, but also the spectral distribution of the emitted radiation. Experiments have shown good agreement with the prediction of equation (1) in the visible part of the spectrum, but in the ultraviolet region the measurements indicate decreasing intensities [for a review see Zrelov (1968)]. As pointed out by Collins and Reiling (1938), "It would be expected, however, that at very short wave-lengths a determination of the intensity would result in a deviation from the classical theory in much the same way that the classical theory of Rayleigh–Jeans fails at short wave-lengths."

3. THE THEORY OF BUDINI

Starting from the problem of the energy lost by a relativistic particle in a polarizable medium, Budini (1953) showed that for the calculation of the energy lost by excitation, ionization, and CR it is necessary to take into account the damping constant of the bound electrons of the medium. The following equation was proposed by Budini for the calculation of the energy lost by CR:

$$\frac{dW}{dx} = \frac{e^2}{c^2} \int \exp\left[-\frac{\omega}{v} \beta^2 \rho \,\text{Im}\, \varepsilon(\omega)\right] \left(1 - \frac{\text{Re}\, \varepsilon(\omega)}{\beta^2 |\varepsilon(\omega)|^2}\right) \omega \, d\omega \tag{4}$$

where the integration limits are defined by the inequality

$$\beta^2 \,\text{Re}\, \varepsilon(\omega) > 1 \tag{5}$$

Re $\varepsilon(\omega)$ and Im $\varepsilon(\omega)$ represent, respectively, the real and the imaginary parts of the dielectric function $\varepsilon(\omega)$, expressed as

$$\varepsilon(\omega) = 1 + \frac{4\pi N e^2}{m} \sum_{k=1}^{r} \frac{f_k}{\omega_k^2 - \omega^2 - i g_k \omega} \tag{6}$$

in which f_k, ω_k, and g_k are the oscillator strength, frequency, and damping constant of the kth absorption limit.

According to Budini, the spectral distribution of the CR can be calculated by use of equations (4) and (5), taking into account the geometrical characteristics of the applied apparatus (ρ denotes the distance from the particle track), and taking into account the dispersion [Re $\varepsilon(\omega)$] and absorption [Im $\varepsilon(\omega)$] characteristics of the medium within which the radiation is generated. The inequality (5) is taken as the condition for the formation of CR; the condition for the CR of a given frequency to be observed at a distance ρ from the track axis is considered to be $\beta^2(\omega/v)\ \mathrm{Im}\ \varepsilon(\omega)\ \rho \ll 1$. Equation (4) reduces to the classical equation (1) in the case of zero damping ($g_k = 0$), i.e., in the ideal case of a perfectly transparent medium without absorption [$\mathrm{Im}\ \varepsilon(\omega) = 0$].

Although this model is very attractive and seems to be a realistic description of CR (the UV divergence in principle is avoided), to recover the threshold velocity and the conical character of radiation (expressed as $\cos\theta = 1/\beta n$, where θ is the emission angle), and also to plot the spectrum of CR, from Budini's theory we will exploit only the condition (5).

4. AN EQUATION FOR THRESHOLD VELOCITY AND SPECTRUM CALCULATION

In the most simple terms, the production of CR is usually explained as follows. The atoms of a dielectric medium in the immediate vicinity of a passing charged particle are distorted by the electric field of the latter into an elongated shape with a nonhomogeneous charge distribution, so they behave as electric dipoles. This polarization is clearly visualized in Figure 1 of Jelley's (1958) book (see also Marmier and Sheldon, 1969). For a slow particle the polarization is symmetrical, hence no electromagnetic radiation is emitted, and for a fast particle with a velocity from a threshold value there results an asymmetrical polarization, giving rise to a resultant dipole field and the emission of CR. If the local polarization vector $\mathbf{P}(\mathbf{r}, t)$ at the point $\mathbf{r} = \mathbf{v}t$ satisfies the equation

$$\frac{d^2\mathbf{P}}{dt^2} + g\frac{d\mathbf{P}}{dt} + \omega_0^2\mathbf{P} = f_0\frac{Ne^2}{m}\mathbf{E}$$

($\mathbf{E}$ is the electric field at $\mathbf{r} = \mathbf{v}t$, g is the coefficient of the friction force, f_0 is the oscillator strength coefficient, and ω_0 is the atomic eigenfrequency without the perturbation), then the Fourier components $\mathbf{E}_\omega$ and $\mathbf{P}_\omega$ with time dependence $\exp(-i\omega t)$ satisfy the equality

$$\mathbf{P}_\omega = \frac{Ne^2}{m}\frac{f_0}{\omega_0^2 - \omega^2 - ig\omega}\mathbf{E}_\omega$$

The electric induction $\mathbf{D}_\omega = \mathbf{E}_\omega + 4\pi\mathbf{P}_\omega$ becomes

$$\mathbf{D}_\omega = \left(1 + \frac{4\pi Ne^2}{m}\frac{f_0}{\omega_0^2 - \omega^2 - ig\omega}\right)\mathbf{E}_\omega$$

or in a familiar form $\mathbf{D}_\omega = \varepsilon(\omega)\mathbf{E}_\omega$. The $\varepsilon(\omega)$ satisfies the conditions described in the Appendix; it differs from the expression (6), but for small frequencies the description of the dielectric properties with dispersion oscillators of one frequency only is a sufficiently good approximation. In practice, for the index of refraction, in general one assumes the form of equation (3); e.g., for a Pilot 425 Cherenkov radiator (Ahlen *et al.*, 1976)

$$a = 1.931 \times 10^{32}/\text{sec}^2 \qquad \text{and} \qquad \omega_0^2 = 4.044 \times 10^{32}/\text{sec}^2$$

By the above picture of the CR phenomenon, it makes sense to attempt to connect the threshold velocity for the formation of CR to the polarization in the particle's vicinity.

A simple calculation show that $\operatorname{Re} \varepsilon(\omega)$ has a maximum at $\omega_M^2 = \omega_0^2 - g\omega_0$. It can be seen that the inequality (5) is satisfied if $1/\beta^2$ is smaller than the maximum value $\operatorname{Re} \varepsilon(\omega_M)$, so $\operatorname{Re} \varepsilon(\omega_M)$ can be related to the threshold velocity by

$$1 + \frac{a_0}{2g\omega_0 - g^2} = \frac{1}{\beta_{\text{thr}}^2} \tag{7}$$

where we introduce the notation $a_0 = 4\pi Ne^2 f_0/m$.

If we take $f_0 \approx 1$ and the atomic density and eigenfrequency are known by assuming β_{thr} as an experimental datum, then from equation (7) we can calculate the damping constant. As the threshold velocity ratio approaches unity, the order of magnitude of g has the order of magnitude of ω_0; in other words, in the resonance process the lifetime of an "elongated atom" is about ω_0^{-1} sec. If it is assumed that for small frequencies, including the visible part, the square of the index of refraction as given in equation (3) equals $\operatorname{Re} \varepsilon(\omega)$, then the constant a_0 may be calculated, too.

As an example, let us consider the CR to be produced in water with $\beta_{\text{thr}} = 0.75$, $n \approx 1.33$ in the visible region, and $\omega_0 = 6 \times 10^{15}\ \text{sec}^{-1}$ (Jelley, 1958). Then we get the solutions (a) $g \approx 4 \times 10^{15}\ \text{sec}^{-1}$, $a_0 \approx 24 \times 10^{30}\ \text{sec}^{-2}$, and (b) $g \approx 5 \times 10^{15}\ \text{sec}^{-1}$, $a_0 \approx 27 \times 10^{30}\ \text{sec}^{-2}$. The $\operatorname{Re} \varepsilon(\omega)$ corresponding to the solution (a) is shown in Figure 1.

The radiated energy will be derived in the next section, with a method which allows us to interpret the resulting expression as the energy lost by the particle in its immediate vicinity, and then radiated by the mechanism described above. Note that in a derivation of equation (1), Nag and Sayied (1956) pointed out that the effect of absorption is eliminated, and the real emitted radiation is obtained. We consider the emitted radiation intensity to be a real observable quantity in a perfectly transparent medium with the

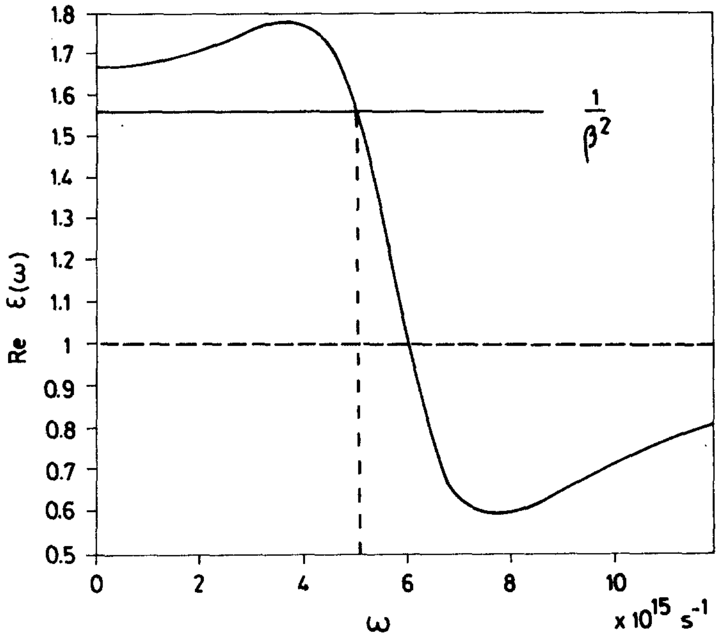

Fig. 1. Plot of Re $\varepsilon(\omega)$ versus ω with the parameters $a_0 = 23.76 \times 10^{30}\ \text{sec}^{-2}$, $g = 3.73 \times 10^{15}\ \text{sec}^{-1}$, and $\omega_0 = 6 \times 10^{15}\ \text{sec}^{-1}$.

index of refraction (3). The spectral energy distribution is written as

$$I(\omega) = \text{const} \cdot \left(\beta^2 - \frac{1}{\text{Re}\ \varepsilon(\omega)}\right)\omega \tag{8}$$

If we choose $\beta = 0.8$ in the preceding example, we get the spectrum as in Figure 2. Note that the solution (b) gives in fact a curve with the same aspect.

5. IONIZATION AND CHERENKOV RADIATION

The ionization energy loss by a fast-moving charged particle in matter is affected by the polarization of the medium (density effect). Fermi (1940) discussed the density effect in detail based on classical electrodynamics. The amount of energy loss by the particle at distances greater than a certain minimum distance b from the path of the particle (b larger than the interatomic distances) was calculated as the flux of the Poynting vector across a cylindrical surface of radius b, having the path of the particle as axis. Fermi found that the ionization loss reached a plateau value at very high energy. The theoretical and experimental aspects of the density effect were reviewed

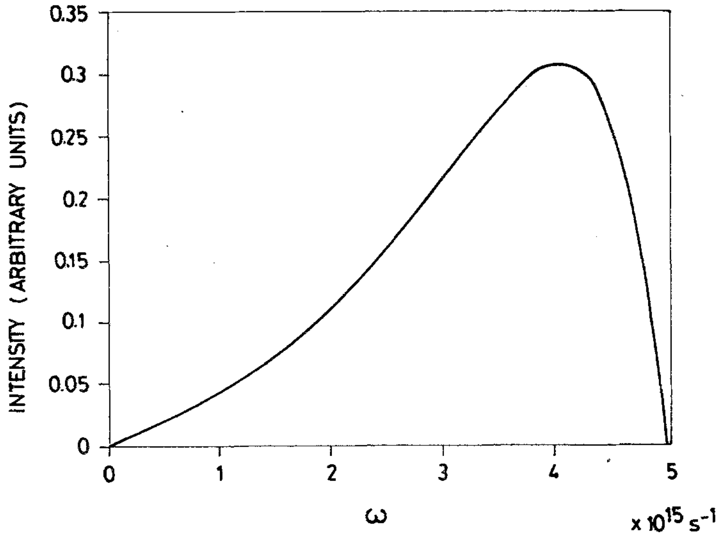

Fig. 2. Theoretical Cherenkov spectrum of radiation induced in water ($n=1.33$, $\beta_{\text{thr}}=0.75$). The velocity ratio is taken to be $\beta=0.8$.

by Crispin and Fowler (1970) and later by Ahlen (1980). The discussion of the classical density effect can also be found in the book of Landau and Lifshitz (1960). Their approach is called semiclassical (Ahlen, 1980) because the distant collisions are treated from the point of view of classical electrodynamics, but it is possible to interpret the vector **k** which appears in the Fourier transform of the fields as the wave vector of an exchanged photon. According to Landau and Lifshitz, one calculates the work done by the particle against the electric field it generates to express the energy loss by CR, too.

For completeness, we repeat the derivation of the general expression of the energy loss [equation (11) below] following Landau and Lifshitz (1960), including the convenient evaluation of the radiated power density by the method of Bornatici and Spada (1989).

Taking into account the relation (A2) of the Appendix, the Maxwell equations are written as

$$\text{rot}\,\mathbf{H}=\frac{1}{c}\frac{\partial\hat{\varepsilon}\mathbf{E}}{\partial t}+\frac{4\pi}{c}\mathbf{j}$$

$$\text{rot}\,\mathbf{E}=-\frac{1}{c}\frac{\partial\mathbf{H}}{\partial t}$$

$$\text{div}\,\hat{\varepsilon}\mathbf{E}=4\pi\rho$$

$$\text{div}\,\mathbf{H}=0$$

where the field is produced by a pointlike source of charge e, moving along the x axis. Then the charge density and the current density can be expressed as

$$\rho=e\,\delta(\mathbf{r}-\mathbf{v}t) \qquad \text{and} \qquad \mathbf{j}=e\mathbf{v}\,\delta(\mathbf{r}-\mathbf{v}t)$$

The electromagnetic potentials $\mathbf{A}$ and φ are introduced in the usual manner,

$$\mathbf{H}=\text{rot}\,\mathbf{A}$$

$$\mathbf{E}=-\,\text{grad}\,\varphi-\frac{1}{c}\frac{\partial\mathbf{A}}{\partial t}$$

By imposing the "generalized" Lorentz condition

$$\text{div}\,\mathbf{A}+\frac{1}{c}\frac{\partial\hat{\varepsilon}\varphi}{\partial t}=0$$

from the Maxwell equations it follows that

$$\nabla^2\mathbf{A}-\frac{\hat{\varepsilon}}{c^2}\frac{\partial^2\mathbf{A}}{\partial t^2}=-\frac{4\pi}{c}e\mathbf{v}\,\delta(\mathbf{r}-\mathbf{v}t) \tag{9a}$$

$$\hat{\varepsilon}\left(\nabla^2\varphi-\frac{\hat{\varepsilon}}{c^2}\frac{\partial^2\varphi}{\partial t^2}\right)=-4\pi e\,\delta(\mathbf{r}-\mathbf{v}t) \tag{9b}$$

Taking the Fourier developments of the potentials

$$\mathbf{A}=\int_{-\infty}^{\infty}\mathbf{A}_k\exp(i\mathbf{k}\cdot\mathbf{r})\,d^3k, \qquad \varphi=\int_{-\infty}^{\infty}\varphi_k\exp(i\mathbf{k}\cdot\mathbf{r})\,d^3k$$

we have for equations (9) the form

$$k^2\mathbf{A}_k+\frac{\hat{\varepsilon}}{c^2}\frac{\partial^2\mathbf{A}_k}{\partial t^2}=\frac{e\mathbf{v}}{2\pi^2c}\exp(-i\mathbf{k}\cdot\mathbf{v}t) \tag{10a}$$

$$\hat{\varepsilon}\left(k^2\varphi_k+\frac{\hat{\varepsilon}}{c^2}\frac{\partial^2\varphi_k}{\partial t^2}\right)=\frac{e}{2\pi^2}\exp(-i\mathbf{k}\cdot\mathbf{v}t) \tag{10b}$$

As can be seen, the Fourier components are dependent on time through the factor $\exp(-\mathbf{ik}\cdot\mathbf{v}t)$. Introducing the notation

$$\omega=\mathbf{k}\cdot\mathbf{v}=k_xv$$

from equations (10) we have

$$\mathbf{A}_k = \frac{e}{2\pi^2 c} \frac{\mathbf{v}}{k^2 - \omega^2 \varepsilon(\omega)/c^2} e^{-i\omega t}$$

and

$$\varphi_k = \frac{e}{2\pi^2 \varepsilon(\omega)} \frac{1}{k^2 - \omega^2 \varepsilon(\omega)/c^2} e^{-i\omega t}$$

The Fourier components of the field are

$$\mathbf{E}_k = \frac{i\omega}{c} \mathbf{A}_k - i\mathbf{k}\varphi_k$$

$$\mathbf{H}_k = i\mathbf{k} \times \mathbf{A}_k$$

and the electric field is

$$\mathbf{E}(\mathbf{r}, t) = \frac{ie}{2\pi^2} \int_{-\infty}^{\infty} \left(\frac{\omega \mathbf{v}}{c^2} - \frac{\mathbf{k}}{\varepsilon(\omega)} \right) \times \frac{1}{k^2 - \omega^2 \varepsilon(\omega)/c^2} \exp[i(\mathbf{k} \cdot \mathbf{r} - \omega t)]\, d^3k$$

The work (or the power density) done by the current against the electric field it generates can be obtained as

$$\frac{dW}{dt} = -\int_{-\infty}^{\infty} \mathbf{j}(\mathbf{r}, t) \cdot \mathbf{E}(\mathbf{r}, t)\, d^3r = -e\mathbf{v} \cdot \mathbf{E}(\mathbf{v}t, t)$$

(Bornatici and Spada, 1989).

On introducing cylindrical coordinates (q, k_x, Φ), with $q^2 = k_y^2 + k_z^2$, and replacing $dk_y\, dk_z$ by $2\pi q\, dq$, we have that the energy loss by the particle per unit path length $(F = dW/dx)$ takes the form

$$F = \frac{ie^2}{\pi} \iint_{-\infty}^{\infty} \frac{[1/\varepsilon(\omega)v^2 - 1/c^2]\omega q}{q^2 + \omega^2[1/v^2 - \varepsilon(\omega)/c^2]}\, dq\, d\omega \qquad (11)$$

In the evaluation of the ionization energy loss the variable q is taken as real, and the integration in equation (11) is carried out from 0 to q_0; $\hbar q_0$ may be interpreted as the maximum momentum transfer.

Now we give the results of Landau and Lifshitz (1960), without repeating the derivations. If $v^2 < c^2/\varepsilon_0$, where $\varepsilon_0 = \varepsilon(0)$, the expression for ionization energy loss is

$$F(q_0) = \frac{4\pi Ne^4}{mv^2}\left(\ln\frac{q_0 v}{\bar{\omega}} + \frac{1}{2}\ln\frac{1}{1-\beta^2} - \frac{\beta^2}{2}\right) \tag{12}$$

($\bar{\omega}$ is the average atomic frequency), and when $v^2 > c^2/\varepsilon_0$ or $v \approx c$ we have the plateau

$$F(q_0) = \frac{2\pi Ne^4}{mc^2}\ln\frac{mc^2 q_0^2}{4\pi Ne^2} \tag{13}$$

These equations are very similar to equations (30) and (31) in Fermi (1940). It should be noted, however, that equations (30) and (31) in Fermi's paper are valid in the case of negligible damping and equations (12) and (13) here can be regarded as the energy loss expressions calculated with zero damping, too.

The starting point in the evaluation of the integrals of equation (11) is the observation that the equation

$$q^2 = \frac{\omega^2}{c^2}\left[\varepsilon(\omega) - \frac{1}{\beta^2}\right] \tag{14}$$

has only one root for $q^2 > 0$, and this root is on the imaginary axis of ω (Landau and Lifshitz, 1960). It can be seen that a root $\omega = i\omega''$ must satisfy the condition $\varepsilon(i\omega'') < 1/\beta^2$, which should hold for arbitrary β, taking into account the properties of $\varepsilon(\omega)$ [see the Appendix, point (c)].

Let us assume now that equation (14) holds for

$$\varepsilon(i\omega'') > \frac{1}{\beta^2} \tag{15}$$

and we take from the beginning q as a complex variable. The integration over ω is carried out for frequencies for which $\varepsilon(\omega)$ is real (i.e., for pure imaginary ω), and the condition (15) is satisfied. Then we have from equation (11)

$$\frac{dW}{dx} = \frac{e^2}{c^2}\frac{1}{2\pi i}\int\left(1 - \frac{1}{\beta^2\varepsilon(i\omega'')}\right)(i\omega'')\,d(i\omega'') \times \int_C \frac{2q\,dq}{q^2 + a^2} \tag{16}$$

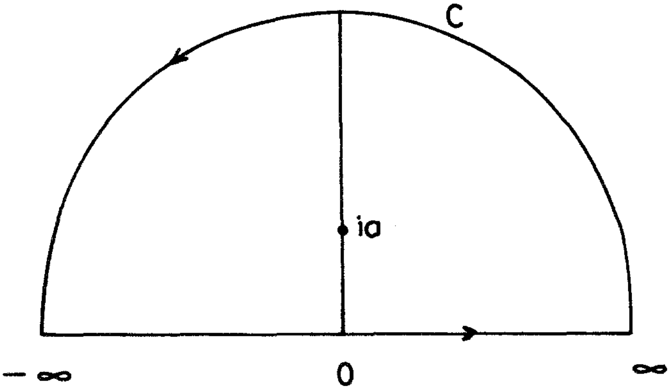

Fig. 3. Contour of integration in the q plane.

where we used the notation

$$a^2 = \frac{(\omega'')^2}{c^2}\left[\varepsilon(i\omega'') - \frac{1}{\beta^2}\right]$$

and the contour of integration C is presented in Figure 3. Then

$$\frac{1}{2\pi i}\int_C \frac{d(q^2+a^2)}{dq}\,\frac{dq}{q^2+a^2} = 1$$

Taking into account the relation (A5) in the Appendix, $\varepsilon(i\omega'')$ may be replaced by Re $\varepsilon(i\omega'')$, and to get a physical result, the complex variable is changed back into a real one. Then equation (16) can be rewritten in the form

$$\frac{dW}{dx} = \frac{e^2}{c^2}\int\left(1 - \frac{1}{\beta^2 \operatorname{Re} \varepsilon(\omega)}\right)\omega\, d\omega \tag{17}$$

with the integration limits determined by the inequality (15) after the above replacements, i.e.,

$$\beta^2 \operatorname{Re} \varepsilon(\omega) > 1 \tag{18}$$

We interpret equation (17) as the energy absorbed by the atoms in the vicinity of the particle and then radiated in the form of observable CR if the inequality (18) is fulfilled.

The angle of emission of the radiation can be obtained from the relation $\cos\theta = k_x/k$, where $k_x = \omega/v$, and $k = \omega/c'$ (c' is the velocity of the emitted light in the medium). Introducing the refractive index as $n(\omega)=c/c'(\omega)$, we recover the classical relation

$$\cos\theta(\omega)=\frac{1}{\beta n(\omega)}$$

which defines the cones of angular aperture $\theta(\omega)$ (Jelley, 1958; Zrelov, 1968). In the limit of vanishingly small absorption, using the dispersion relation (3), and Re $\varepsilon(\omega)$ from the preceding section, it is obvious that the inequality $\beta n(\omega)>1$ is a natural consequence of the inequality (18). This can be seen easily from the example of Figure 1.

6. CONCLUDING REMARKS

The energy loss of a pointlike charged particle by ionization and CR has been evaluated by means of the Fourier transform method. It has been shown that in the CR process the polarization in the immediate vicinity of the passing particle can be described with a damping constant which has the magnitude of the atomic eigenfrequency, as results from equation (7). In this way it is possible to avoid the UV divergent spectrum by the use of equation (17). The angle of emission of CR remains the same as in the classical theory, but the usual inequality (2) of CR theory appears to be a consequence of the inequality (18), and the light intensity at angle $\theta(\omega)$ is given by equation (8).

A quantum electrodynamic discussion of the ionization and CR, based on the electron self-energy (mass operator) method, will be given in a later paper.

APPENDIX

In this Appendix we give a short review of the dispersion properties of the dielectric function, following Landau and Lifshitz (1960).

In a continuous isotropic medium the most general relation between the electric induction $\mathbf{D}(t)$ and the field $\mathbf{E}(t)$ can be written as

$$\mathbf{D}(t)=\mathbf{E}(t)+\int_0^{\infty} f(\tau)\mathbf{E}(t-\tau)\,d\tau \qquad \text{(A1)}$$

where $f(\tau)$ is a function of time and depends on the medium's properties. In the range of integration, $f(\tau)<\infty$. Equation (A1) can be put in the form

$$\mathbf{D}(t)=\hat{\varepsilon}\mathbf{E}(t) \qquad \text{(A2)}$$

where $\hat{\varepsilon}$ denotes a linear integral operator.

If the Fourier components of the quantities $\mathbf{D}(t)$ and $\mathbf{E}(t)$ depend on time through the factor $\exp(-i\omega t)$, then equation (A2) becomes

$$\mathbf{D}_\omega = \varepsilon(\omega)\mathbf{E}_\omega \tag{A3}$$

where $\varepsilon(\omega)$ is a function given by the equation

$$\varepsilon(\omega) = 1 + \int_0^\infty f(\tau)\, e^{i\omega\tau}\, d\tau \tag{A4}$$

which can be written by separating the real and the imaginary parts as

$$\varepsilon(\omega) = \text{Re } \varepsilon(\omega) + i \text{ Im } \varepsilon(\omega)$$

By taking ω as complex variable, $\omega = \omega' + i\omega''$, we establish the following properties of the function $\varepsilon(\omega)$:

(a) $\varepsilon(\omega)$ is defined in the upper half-plane of ω. If $\omega'' < 0$, the integral in equation (A4) is divergent.

(b) $\varepsilon(\omega)$ is a real function only for pure imaginary ω. As can be seen from equation (A4), $\varepsilon(i\omega'') = \varepsilon^*(i\omega'')$ (the asterisk denotes complex conjugation), i.e.,

$$\varepsilon(i\omega'') = \text{Re } \varepsilon(i\omega'') \tag{A5}$$

(c) On the imaginary axis $\varepsilon(\omega)$ is monotonically decreasing from the value $\varepsilon(i0) > 1$ to $\varepsilon(i\infty) = 1$.

(d) The real part and the imaginary part of $\varepsilon(\omega)$ are related by the Kramers–Kronig dispersion relations,

$$\text{Re } \varepsilon(\omega) = 1 + \frac{1}{\pi} P \int_{-\infty}^{\infty} \frac{\text{Im } \varepsilon(\omega)}{x - \omega}\, dx$$

$$\text{Im } \varepsilon(\omega) = -\frac{1}{\pi} P \int_{-\infty}^{\infty} \frac{\text{Re } \varepsilon(\omega) - 1}{x - \omega}\, dx$$

where x and ω are real variables, and P means principal part.

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